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BOUNDED HARMONIC FUNCTIONS ON NONAMENABLE COVERS OF COMPACT MANIFOLDS

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ABSTRACT
We show that the space of bounded harmonic functions on a nonamenable cover of a compact Riemannian manifold is infinite dimensional.

1. Introduction

The study of harmonic functions on Riemannian manifolds, i.e. the solutions of the equation $\Delta h = 0$ where Δ is the Laplace–Beltrami operator, has attracted much attention recently. There is a satisfactory description of spaces of bounded and positive harmonic functions on negatively curved manifolds in terms of the Poisson and Martin boundaries (see Anderson and Shoen [AS], Ancona [A], and Kifer [K]). In this note we shall study bounded harmonic functions on a connected cover M of a smaller Riemannian manifold N, i.e. $N = M/\Gamma$ for some discrete group Γ of isometries acting on M. Recall that a discrete group Γ is called amenable if there is a finitely additive, translation invariant nonnegative probability measure defined for all subsets of Γ. A cover M of N is called nonamenable if the group Γ in the representation $N = M/\Gamma$ is nonamenable.

THEOREM A (Lyons and Sullivan [LS]). *Any nonamenable cover M of any Riemannian manifold N possesses nonconstant bounded harmonic functions.*

Lyons and Sullivan's idea was to choose an invariant mean φ for the abelian

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additive semigroup $R_+ = \{t : t \geqq 0\}$ and apply it to the function $P_t f(x) = \int_M p(t, x, y) f(y) dm(y)$ thought as a bounded continuous function on R_+ for each x, where f is a bounded function on M, $p(t, x, y)$ is the heat kernel (i.e. the transition density of the Brownian motion on M), and m denotes the Riemannian volume. Next, one shows that $\varphi(P_t f)(x)$ is a harmonic function and if it is constant for any bounded f then $\varphi(P_t \cdot)$ generates an invariant mean on Γ which is impossible for a nonamenable Γ.

The following result describes an important class of nonamenable covers.

THEOREM B (Ballman and Eberlein [BE]). *Let N be a Riemannian manifold with finite volume all of whose sectional curvatures are nonpositive and bounded from below by a constant $-a^2$. Then either N is flat or its fundamental group Γ contains a nonabelian free subgroup, and so (see, for instance, Introduction to* [LS]) *in the second case Γ is nonamenable, i.e. the universal cover M of N ($N = M/\Gamma$) is nonamenable.*

The goal of this note is to prove

THEOREM C. *Let M be a cover of a compact Riemannian manifold $N = M/\Gamma$. If M admits a nonconstant bounded harmonic function, then the linear space of such functions is infinite dimensional.*

Theorems A–C yield

COROLLARY. *Any nonamenable cover of a compact Rimannian manifold, in particular, the universal cover of a nonflat compact manifold of nonpositive curvature, possesses an infinite-dimensional space of bounded harmonic functions.*

These imply also that in the above cases the Poisson and Martin boundaries of M are infinite sets.

2. Auxiliary lemmas

We shall start with the following simple fact.

LEMMA 1. *Let $h_1, \ldots, h_n$ be linearly independent functions on some space M. Then there exist n points $x_1, \ldots, x_n$ such that n-vectors $\bar{h}(x_i) = (h_1(x_i), \ldots, h_n(x_i))$, $i = 1, \ldots, n$ are linearly independent.*

PROOF. For any finite set of points $\Gamma = \{y_1, \ldots, y_m\} \subset M$ denote by $\mathscr{A}_\Gamma$ the set of unit n-vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\Sigma_{i=1}^n \alpha_i h_i(y_k) = 0$ for all

$k = 1, \ldots, m$. Clearly, each $\mathscr{A}_\Gamma$ is a closed subset of the $(n-1)$-dimensional unit sphere S^{n-1}. If $\mathscr{A}_\Gamma$ is empty then the rank of the $n \times m$ matrix

$$\begin{bmatrix} h_1(y_1) & \cdots & h_1(y_m) \\ \vdots & & \vdots \\ h_n(y_1) & \cdots & h_n(y_m) \end{bmatrix}$$

equals n, and so one can choose n linearly independent columns proving the lemma. Suppose that, on the contrary, the sets $\mathscr{A}_\Gamma$ are not empty for all finite subsets Γ of M. Since all sets $\mathscr{A}_\Gamma$ are closed subsets of the compact space S^{n-1} and

$$\mathscr{A}_{\Gamma_1} \cap \mathscr{A}_{\Gamma_2} \cap \cdots \cap \mathscr{A}_{\Gamma_l} = \mathscr{A}_{\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_l} \neq \varnothing$$

for any finite collection of finite subsets $\Gamma_i \subset M$, $i = 1, \ldots, l$, then it follows that the intersection of all $\mathscr{A}_\Gamma$ is not empty. But if $\alpha = (\alpha_1, \ldots, \alpha_n)$ belongs to this intersection, then $\Sigma_{i=1}^n \alpha_i h_i(y) = 0$ for all $y \in M$ which contradicts the linear independence of h_i, $i = 1, \ldots, n$. □

We shall need another well-known algebraic fact.

Lemma 2. *Let Q be a subgroup of the matrix group* $\mathrm{GL}(n, R)$ *such that, for some collection of n linearly independent n-vectors $\xi_1, \ldots, \xi_n$, the set $\{q\xi_i, q \in Q, i = 1, \ldots, n\}$ is bounded. Then Q is conjugate to a subgroup of the group $O(n)$ of orthogonal matrices, i.e. for some matrix B, BQB^{-1} is a subgroup of $O(n)$.*

Proof. It follows that Q is a compact subgroup of $\mathrm{GL}(n, R)$, and so it is conjugate to a subgroup of the maximal compact subgroup $O(n)$ (see, for instance, Helgason [H]).

2. Proof of Theorem C

To prove Theorem C suppose that, on the contrary, the space of bounded harmonic functions on M is finite-dimensional but not one-dimensional. Let $f_1 \equiv 1, f_2, \ldots, f_n$; $n > 1$, be a basis of this space. Since all members of the group Γ are isometries and the Laplace–Beltrami operator is invariant under isometries, then for any bounded harmonic function $f(x)$ the function $(T_\gamma f)(x) = f(\gamma x)$, $\gamma \in \Gamma$ is also harmonic and $T_\gamma f$ has the same upper and lower bounds as f. Thus

$$(T_\gamma f_i)(x) = f_i(\gamma x) = \sum_{j=1}^{n} a_{ij}(\gamma) f_j(x)$$

with $a_{11}(\gamma) = 1$ and $a_{1j}(\gamma) = 0$ for $j = 2, \ldots, n$. Using notations

$$\bar{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad \text{and} \quad A_\gamma = (a_{ij}(\gamma))$$

we can write $(T_\gamma \bar{f})(x) = \bar{f}(\gamma x) = A_\gamma \bar{f}(x)$. Then for any $\gamma_1, \gamma_2 \in \Gamma$ one has $A_{\gamma_2\gamma_1}\bar{f} = A_{\gamma_2}A_{\gamma_1}\bar{f}$. By Lemma 1 we conclude that the linear system $(A_{\gamma_2}A_{\gamma_1} - A_{\gamma_2\gamma_1})\xi = 0$ must have n linearly independent solutions

$$\xi^{(i)} = \bar{f}(x_i), \qquad i = 1, \ldots, n,$$

for some points x_i chosen according to Lemma 1. This yields $A_{\gamma_2}A_{\gamma_1} = A_{\gamma_2\gamma_1}$ for any $\gamma_1, \gamma_2 \in \Gamma$, i.e. we obtain a representation of Γ into $GL(n, R)$. Note that

$$\sup_x |A_\gamma \bar{f}(x)| = \sup_x |\bar{f}(\gamma x)| = \sup_x |\bar{f}(x)| = \sup_x \left(\sum_{i=1}^{n} f_i^2(x) \right)^{1/2} < \infty$$

since $f_1, \ldots, f_n$ are bounded, and so by Lemmas 1 and 2 we obtain that the group of matrices $\{A_\gamma, \gamma \in \Gamma\}$ is conjugate to a subgroup of $O(n)$. Hence there exists a matrix $B \in GL(n, R)$ such that $C_\gamma = BA_\gamma B^{-1}$ is an orthogonal matrix for any $\gamma \in \Gamma$. Consider another basis $\bar{g} = B\bar{f}$,

$$\bar{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

of the space of bounded harmonic functions on M. Then $\bar{g}(\gamma x) = C_\gamma \bar{g}(x)$ (and, clearly, we obtain an orthogonal representation of the group Γ).

We claim that one can find an orthogonal matrix $U \in O(n)$ such that the new basis $\bar{h} = U\bar{g}$,

$$\bar{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

of the space of bounded harmonic functions has $h_1(x) \equiv \text{const}$. Indeed, choose points $x_1, \dots, x_n$ according to Lemma 1 so that the vectors $\bar g(x_1), \dots, \bar g(x_n)$ are linearly independent. Let us find $U = (u_{ij}) \in O(n)$ such that $\Sigma_{j=1}^n u_{1j} g_j(x_k)$ equals a constant independent of $k = 1, \dots, n$. To do this we remark that the matrix

$$(\delta_{ij}) = (g_j(x_i)), \qquad i, j = 1, \dots, n$$

satisfies $\det(\delta_{ij}) \neq 0$ since the vectors $\bar g(x_i)$, $i = 1, \dots, n$ are linearly independent. Thus the system $\Sigma_{j=1}^n \delta_{ij} \xi_j = 1$, $i = 1, \dots, n$ has the unique solution

$$\bar\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} .$$

Now put

$$u_{1j} = \xi_j \left(\sum_{j=1}^n \xi_j^2 \right)^{-1};$$

then

$$\sum_{j=1}^n u_{1j} g_j(x_k) = \left(\sum_{j=1}^n \xi_j^2 \right)^{-1},$$

for all $k = 1, \dots, n$. This gives the first row of the matrix U. We choose other rows of U so that they complement the vector $(u_{11}, u_{12}, \dots, u_{1n})$ to an orthonormal basis of R^n. Define $\bar h = U \bar g$,

$$\bar h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} .$$

We assert that $h_1(x) \equiv \text{const}$. Indeed, since $h_1\ h_2, \dots, h_n$ is a basis of the space of bounded harmonic functions and 1 belongs to this space, then for some numbers $\alpha_1, \dots, \alpha_n$ we can write $\Sigma_{i=1}^n \alpha_i h_i(x) \equiv 1$. On the other hand, by our construction $h_1(x_k) = C$ for some constant $C > 0$ and all x_k, $k = 1, \dots, n$ chosen above. Then

$$(C\alpha_1 - 1) h_1(x_k) + \sum_{i=2}^n C\alpha_i h_i(x_k) = 0 \quad \text{for all } k = 1, 2, \dots, n.$$

Since the vectors $\bar{h}(x_k)$, $k = 1, \ldots, n$ are linearly independent, then the vectors $(h_i(x_1), h_i(x_2), \ldots, h_i(x_n))$, $i = 1, \ldots, n$ are also linearly independent, and so the above equality yields $\alpha_1 = C^{-1}$, $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$. Hence $h_1(x) \equiv C$.

Next, we have $\bar{h}(\gamma x) = U\bar{g}(\gamma x) = UC_\gamma \bar{g}(x) = V_\gamma \bar{h}(x)$ where $V_\gamma = UC_\gamma U^{-1}$ is an orthogonal matrix. Since $h_1(\gamma x) = h_1(x) \equiv C$, then we conclude that the first row of any matrix V_γ must have one in the first column and zero in all other columns. Since V_γ is orthogonal, this means that it has the block form

$$V_\gamma = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & W_\gamma \\ 0 & \end{bmatrix}$$

where W_γ is an $(n-1) \times (n-1)$ orthogonal matrix. Thus denoting

$$\bar{h}^{(1)} = \begin{bmatrix} h_2 \\ \vdots \\ h_n \end{bmatrix}$$

we obtain $\bar{h}^{(1)}(\gamma x) = W_\gamma \bar{h}^{(1)}(x)$. Note that no linear combination of the functions $h_2, \ldots, h_n$ can be identically equal to a constant other than zero, since otherwise $h_1, h_2, \ldots, h_n$ would be linearly dependent.

For any vector $\bar{\alpha} = (\alpha_2, \ldots, \alpha_n)$ with $|\alpha|^2 = \Sigma_{i=2}^n \alpha_i^2 = 1$ and $x \in M$ define the function

$$F(\alpha, x) = \sum_{i=2}^{n} \alpha_i h_i(x) = (\bar{\alpha}, \bar{h}^{(1)}(x)).$$

We have $F(\alpha, \gamma x) = F(W_\gamma^* \alpha, x) = F(W_\gamma^{-1}\alpha, x)$ and $|W_\gamma^{-1}\alpha| = 1$ since $W_\gamma \in O(n-1)$. Thus if $\tilde{N}$ is a compact fundamental domain of the group Γ, then

$$L_{\max} = \sup_{\alpha : |\alpha| = 1, x \in \tilde{N}} |F(\alpha, x)| = \sup_{\alpha : |\alpha| = 1, x \in M} |F(\alpha, x)|.$$

Since F is continuous, then there exists a pair $(\alpha^{(0)}, x_0)$, $|\alpha^{(0)}| = 1$, $x_0 \in \tilde{N}$ such that $F(\alpha^{(0)}, x_0) = L_{\max}$.

Next, consider the harmonic function $h^{(0)}(x) = \Sigma_{i=2}^n \alpha_i^{(0)} h_i(x)$. Let $p(t, x, y)$ be the heat kernel on M (see Chavel [C]), i.e. the minimal positive fundamental solution of the equation $\partial p/\partial t = \Delta_x p$. Recall that from the probabilistic point of view $p(t, x, y)$ is the transition density of the Brownian motion on M. Note that since the metric on M is lifted from the compact manifold N, then the curvature of M is bounded, and so (see Yau [Y] or Ikeda and Watanabe [IW], p. 381) $\int_M p(t, x, y)dm(y) = 1$, where m denotes the Riemannian volume on M. This means that the Brownian motion on M has no explosions. As for any harmonic function, the harmonic function $h^{(0)}$ defined above satisfies (see, for instance, Dynkin [D])

$$h^{(0)}(x) = \int p(t, x, y)h^{(0)}(y)dm(y).$$

Since

$$h^{(0)}(x_0) = \sup_{y \in M} h^{(0)}(y) = L_{\max}, \quad p(t, x, y) > 0, \quad \int p(t, x, y)dm(y) = 1,$$

and $h^{(0)}(x)$ is continuous, it follows that $h^{(0)}(x)$ equals $L_{\max}$ identically. This is a contradiction because $h^{(0)}$ is a linear combination of $h_2, h_3, \ldots, h_n$, and so must be independent of $h_1 \equiv \text{const}$, completing the proof of Theorem C.

4. Concluding remarks

In fact, we have proved the following general result.

THEOREM C′. *Let Γ be a discrete group acting on a separable metric space M so that the factor M/Γ is compact. Suppose that H is a linear subspace of the space of all bounded continuous functions on M such that no nonconstant function from H may attain its supremum in a point of M. If H is invariant under the action of all operators T_γ, $\gamma \in \Gamma$ given by $T_\gamma f(x) = f(\gamma x), f \in H, x \in M$, then either H contains only constants or H is infinite-dimensional.*

It is clear that Theorem C (as well as Theorem C′) is not true without the compactness assumption on $N = M/\Gamma$, since otherwise we can take $M = N$ to be a noncompact Riemannian manifold possessing only finite-dimensional (but not one-dimensional) space of bounded harmonic functions. Consider, for instance, the following simple example, where $M = N = R^1$ is the real line with the metric $ds = e^{-x^2/2}dx$, i.e.

$$\text{dist}(a, b) = \int_a^b e^{-x^2/2}dx \quad \text{for } b \geqq a.$$

Then we shall have the finite "volume" Riemannian manifold with the Laplace–Beltrami operator

$$\Delta = e^{x^2/2}\frac{d}{dx}\left(e^{x^2/2}\frac{d}{dx}\right) = e^{x^2}\left(\frac{d^2}{dx^2} + x\frac{d}{dx}\right).$$

Hence the harmonic functions will be the solutions of the equation

$$\frac{d^2h(x)}{dx^2} + x\frac{dh(x)}{dx} = 0.$$

It is easy to see that the space of bounded harmonic functions here is two-dimensional. From the probabilistic point of view this reflects the fact that the corresponding transient diffusion (called in this case the Ornstein–Uhlenbeck process) may approach infinity by two ways: going either to ∞ or to $-\infty$. The analytic argument is also simple. Representing a solution as a power series $h(x) = \Sigma_{k=0}^{\infty} a_k x^k$ and substituting it in the above equation, we see that a_0 and a_1 can be chosen arbitrarily, $a_{2k} = 0$ for all $k = 1, 2, \ldots$, and the odd coefficients satisfy

$$a_{2k+1} = -\frac{(2k-1)a_{2k-1}}{2k(2k+1)} = (-1)^k \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2k-1)}{(2k+1)!} a_1.$$

Thus we shall obtain that the space of solutions is two-dimensional with the basis $h_1 \equiv 1$ and $h_2(x) = \int_0^x e^{-y^2/2}dy$.

The above example does not contradict, however, a conjecture that we may omit the compactness assumption in the first part of Corollary, namely, that any nonamenable cover of any Riemannian manifold (not necessarily compact) possesses an infinite-dimensional space of bounded harmonic functions. This conjecture appeals to the common sense that on a "large" manifold with a "complicated" isometry group there exist a lot of harmonic functions, since they can be described in terms of "exits to infinity" of the Brownian motion.

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Remark. The referee drew my attention to the preprint of H. Donnelly, *Bounded harmonic functions and positive Ricci curvature*, Math. Z. **191** (1986), 559–565, establishing that a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set has only a finite-dimensional space V of bounded harmonic functions and examples with dim $V > 1$ are constructed.

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